

# Spectral representation of the particle production out of equilibrium – Schwinger mechanism in pulsed electric fields

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**Abstract.** We develop a formalism to describe the particle production out of equilibrium in terms of dynamical spectral functions, i.e. Wigner transformed Pauli-Jordan's and Hadamard's functions. We take an explicit example of a spatially homogeneous scalar theory under pulsed electric fields and investigate the time evolution of the spectral functions. In the out-state we find an oscillatory peak in Hadamard's function as a result of the mixing between positive- and negative-energy waves. The strength of this peak is of the linear order of the Bogoliubov mixing coefficient, whereas the peak corresponding to the Schwinger mechanism is of the quadratic order. Between the in- and the out-states we observe a continuous flow of the spectral peaks together with two transient oscillatory peaks. We also discuss the medium effect at finite temperature and density. We emphasise that the entire structure of the spectral functions conveys rich information on real-time dynamics including the particle production.

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## 1. Introduction

Quantum field theory has been quite successful in describing non-trivial contents of the vacuum and the  $S$ -matrix elements from the vacuum-to-vacuum amplitude using the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula. For static quantities the numerical Monte-Carlo simulation of the lattice discretised theory is so powerful that one could perform the first-principle calculation in a non-perturbative way. In contrast to tremendous achievements for static quantities, the numerical machinery for solving the real-time dynamics (or the initial value problem) has not been established beyond the scope of the linear response theory. The point is that one should compute not the vacuum-to-vacuum amplitude but an expectation value at time  $t$ , which involves time-evolution operators from  $-\infty$  to  $t$  and also its Hermite conjugate. It is known as the closed-time path (CTP) formalism [1, 2] how to deal with two time-evolution operators with generalised Green's functions. The microscopic derivation of the Boltzmann

equation was pioneered by Kadanoff and Baym [3] (see also [4, 5] for recent studies), in which the Wigner transform of correlation functions translates to the distribution function and the spectral function.

Generally speaking, the spectral functions provide us with detailed information on physical contents in the system. Even in the case of equilibrated matter the spectral function represents in-medium dispersion relations of collective excitations such as the plasmon, the zero sound, etc (see [6, 7] for classical textbooks). One can even infer the real-time properties near equilibrium by the analytical continuation of Green's functions once a spectral function is available. In this way, using the spectral function (or the imaginary part of the retarded self-energy), one can evaluate the thermal emission rate of a pair of particle and anti-particle (or hole in condensed matter systems) [8, 9]. Such thermal processes are allowed in a medium where thermally excited particles are brought in. In this kind of calculation in equilibrated matter, the translational invariance in time needs not be violated and the ordinary field-theory techniques are useful.

A more non-trivial example of the particle production is the process induced by the presence of time-dependent external field. The pioneering work by Heisenberg and Euler [10] has revealed that the one-loop effective action on top of electromagnetic background fields has an imaginary part. This indicates that the vacuum becomes unstable; in other words, the particle production can occur from the vacuum. The vacuum permittivity has also been formulated in the field-theoretical manner by Schwinger [11]. Named after his seminal work, the pair production of particle and anti-particle from the vacuum under electric field is commonly referred to as the Schwinger mechanism (see [12] for a comprehensive review). This could be regarded as a special example of the Landau-Zener effect (see [13] for example).

The essence for the particle production from the vacuum is concisely represented by the Bogoliubov transformation of the creation/annihilation operators, with which the positive- and the negative-energy states are mixed together. We note that the celebrated Hawking radiation, that is the particle production under gravitational effects, belongs to the same class of physics. In short, the vacuum defined in the “in-state” is filled with particles and anti-particles if seen in the “out-state” where the observer stands, and the Bogoliubov transformation connects the in- and the out-states by a unitary rotation.

The Schwinger mechanism and the Hawking radiation are quantum (tunnelling) phenomena and have been intensively studied in the semi-classical method like the Wentzel-Kramers-Brillouin (WKB) approximation (see [14] for a review, and also [15] for the WKB formulation of the Hawking radiation). Although the mixing via the Bogoliubov transformation is straightforward and the semi-classical methods appeal to our intuition, it would be more desirable to develop a systematic formulation in terms of the field correlators. We believe that this reformulation is indispensable for future progresses; someday one might be able to execute real-time numerical simulations, and then, the canonical quantisation with creation/annihilation operators is not quite compatible with numerical algorithms. Ideally, if we can express the Schwinger mechanism using some spectral functions in analogy with the thermal emission rate, we

could attain a unified view of the particle production near and out of equilibrium.

Some time ago the present author has formulated the Schwinger mechanism in a form similar to the LSZ reduction formula in [16], which is based on preceding works [17, 18]. A variant of this formula is also used in a recent attempt to utilise the classical statistical approximation to simulate the Schwinger mechanism numerically [19]. As we discuss later, though the LSZ-type formula looks reasonable, the treatment of the in-state has some subtlety. If we consider the inclusive spectrum only, in fact, we can easily derive another formula given in terms of the spectral functions without any ambiguity. In this case the translational invariance in time is lost and we should handle the *dynamical (time-dependent) spectral functions*. It should be thus a natural idea to look into the temporal change of the spectral functions in accord with the quasi-particle contents affected by the time-dependent background fields.

We stress that reformulating the problem of the particle production makes an important building block in a timely subject; real-time dynamics is the key issue in various fields of physics. In the research of the quark-gluon plasma formation for instance, the thermalization process is under active dispute (see [20] and references therein). Large laser facilities are aiming to detect the production of a pair of electron and positron and it has been discovered that the dynamically assisted Schwinger mechanism significantly reduces the critical strength of the electric field [21, 22]. For precise theoretical predictions it is strongly demanded to invent a new scheme for the full quantum real-time simulation. Probably, to achieve this goal, the stochastic quantisation is one of the most promising approaches [23]. However, the conventional description of the Schwinger mechanism or the Hawking radiation does not fit in with the functional language with which the stochastic quantisation is written. This highly motivates us to think of the spectral representation of the particle production out of equilibrium, as is the main topic of this work.

In this paper we will first give a detailed account of the derivation of our formula with the spectral functions. Then, we will investigate the general properties of the spectral functions associated with the in- and the out-states involving the Bogoliubov transformation. We can understand that the Schwinger mechanism accesses only a small portion of the whole spectral functions. This means, hence, that the spectral functions contain much more information than the Schwinger mechanism and new possibilities for a better detection might be still buried in them. The dynamical spectral functions thus deserve serious investigations and we will construct them concretely for a special case of homogeneous pulsed electric fields to dig non-trivial features out.

## 2. Particle number out of equilibrium

Let us consider a general setup to formulate the particle production out of equilibrium in quantum field theory. In this paper we focus only on a single-component complex scalar field (i.e. scalar QED [24]) to simplify the expressions, but the generalisation to other fields such as fermions and multi-component fields is not difficult [25].

We require the existence of well-defined asymptotic states, namely, the in-state at  $t = -\infty$  and the out-state at  $t = \infty$ , where the interactions should be turned off. In our convention we put “in” and “out” in the subscript to refer to quantities that belong to the in-state and the out-state, respectively. With increasing time, thus, the energy dispersion relation should evolve from  $p_0 = E_{\text{in}}(\mathbf{p})$  to  $p_0 = E_{\text{out}}(\mathbf{p})$  driven by interactions with external fields, and we would like to compute the particle number associated with this change. For this purpose the expression for the inclusive spectrum is our starting point, which is given by the number operator as

$$\frac{dN}{d^3\mathbf{p}} = \frac{1}{(2\pi)^3} \langle \hat{\rho}_{\text{in}} \hat{n}_{\text{out}}(\mathbf{p}) \rangle := \frac{1}{(2\pi)^3} \langle \hat{\rho}_{\text{in}} \hat{a}_{\text{out}}^\dagger(\mathbf{p}) \hat{a}_{\text{out}}(\mathbf{p}) \rangle. \quad (1)$$

Here  $\hat{\rho}_{\text{in}}$  represents the density matrix that characterises the in-state. If we choose it to be a pure state of *the initial vacuum*,  $\hat{\rho}_{\text{in}} = |0_{\text{in}}\rangle\langle 0_{\text{in}}|$ , there is no contribution to (1) from the initial state. Then (1) counts the number of produced particles only. We make a remark that, if the initial state contains particles, we may utilise (1) to address the problem of particle absorption as well as particle production.

Our goal at the moment is to find an alternative expression of (1) in terms of field variables instead of creation/annihilation operators. To this end we need a prescription to identify creation/annihilation operators under background fields. These operators are related to the field operator  $\hat{\phi}(x)$  via the expansion on complete basis, which is a clean procedure in the asymptotic states. In the out-state the annihilation operator is extracted through

$$\begin{aligned} \sqrt{2E_{\text{out}}(\mathbf{p})} \hat{a}_{\text{out}}(\mathbf{p}) &= \lim_{t \rightarrow \infty} i \int d^3\mathbf{x} e^{iE_{\text{out}}(\mathbf{p})t - i\mathbf{p}\cdot\mathbf{x}} [\partial_t - iE_{\text{out}}(\mathbf{p})] \hat{\phi}(t, \mathbf{x}) \\ &= \lim_{t \rightarrow \infty} i e^{iE_{\text{out}}(\mathbf{p})t} [\partial_t - iE_{\text{out}}(\mathbf{p})] \hat{\phi}(t, \mathbf{p}). \end{aligned} \quad (2)$$

We use the same notation  $\hat{\phi}$  also for the Fourier transformed field as long as no confusion arises. In our convention the normalisation above is consistent with the commutation relation,  $[\hat{a}_{\text{out}}(\mathbf{p}), \hat{a}_{\text{out}}^\dagger(\mathbf{p}')] = (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}')$ . This (2) is a basic relation frequently used in the derivation of the LSZ reduction formula in many textbooks. Because the number operator involves the creation/annihilation operators at the equal time, we can drop the exponential part and simplify the formula as

$$\hat{a}_{\text{out}}^\dagger(\mathbf{p}) \hat{a}_{\text{out}}(\mathbf{p}) = \frac{1}{2E_{\text{out}}(\mathbf{p})} \lim_{t_1=t_2=t \rightarrow \infty} [\partial_{t_1} + iE_{\text{out}}(\mathbf{p})] [\partial_{t_2} - iE_{\text{out}}(\mathbf{p})] \hat{\phi}^\dagger(t_1, \mathbf{p}) \hat{\phi}(t_2, \mathbf{p}). \quad (3)$$

If we are interested in the production of anti-particles, we can find a similar formula replacing  $\hat{a}_{\text{out}}^\dagger(\mathbf{p}) \hat{a}_{\text{out}}(\mathbf{p})$  with  $\hat{b}_{\text{out}}^\dagger(-\mathbf{p}) \hat{b}_{\text{out}}(-\mathbf{p})$ . Owing to the conservation of U(1) charge (electric charge), the number of anti-particles should be anyway identical with that of particles, so we do not calculate it explicitly here.

In view of this form it is already clear that we can translate (1) into a representation by means of the Wightman function [26]. It should be more illuminating to find an alternative expression using the spectral functions instead of the Wightman function, for the spectral functions provide us with more intuition about physical contents of the

system. Let us define the spectral functions or the Wigner transformed Pauli-Jordan's (denoted by  $\mathcal{A}$ ) and Hadamard's (denoted by  $\mathcal{D}$ ) functions as follows;

$$\begin{aligned}\mathcal{A}^\Delta(t, p_0, \mathbf{p}) &:= \frac{1}{V} \int_{-\Delta}^{\Delta} d\delta t e^{ip_0\delta t} \langle \hat{\rho}_{\text{in}} [\hat{\phi}(t + \tfrac{1}{2}\delta t, \mathbf{p}), \hat{\phi}^\dagger(t - \tfrac{1}{2}\delta t, \mathbf{p})] \rangle, \\ \mathcal{D}^\Delta(t, p_0, \mathbf{p}) &:= \frac{1}{V} \int_{-\Delta}^{\Delta} d\delta t e^{ip_0\delta t} \langle \hat{\rho}_{\text{in}} \{ \hat{\phi}(t + \tfrac{1}{2}\delta t, \mathbf{p}), \hat{\phi}^\dagger(t - \tfrac{1}{2}\delta t, \mathbf{p}) \} \rangle.\end{aligned}\quad (4)$$

Here we put a volume factor  $V$  because we look at the same momenta  $\mathbf{p}$  and trivially there arises  $2\pi\delta(0) = V$ . One could define the spectral functions with two momentum arguments, which would be useful in the presence of spatially modulated background fields. In this work, however, we consider only the spatially homogeneous case, so that the above definition (4) suffices for our goal. It should be mentioned that our definition of (4) explicitly depends on an extra parameter  $\Delta$ . In the Wigner transformation, usually, one formally takes  $\Delta \rightarrow \infty$ . For a practical application to the numerical analysis, a finite  $\Delta$  as above would be legitimate, as we will discuss later. To extract information on the in- or the out-state, as a matter of fact, one should keep the ordering,  $|t| \gg \Delta$ , when we formally take  $\Delta \rightarrow \infty$ ; otherwise the spectral functions are affected by the interaction even for  $t$  that is far outside of the interacting region. Roughly speaking,  $\Delta$  should be interpreted as an “observation time” with which the quasi-particle oscillation is resolved.

We can change the variables from  $t_1$  and  $t_2$  to  $t = \frac{1}{2}(t_1 + t_2)$  and  $\delta t = t_1 - t_2$ , so that we can finally arrive at the following formula,

$$\frac{dN}{d^3\mathbf{p}} = \lim_{t \rightarrow \infty} \frac{V}{(2\pi)^3} \int \frac{dp_0}{2\pi} \frac{1}{4E_{\text{out}}(\mathbf{p})} \left[ \frac{1}{4} \partial_t^2 + (p_0 + E_{\text{out}}(\mathbf{p}))^2 \right] [\mathcal{D}^\Delta(t, p_0, \mathbf{p}) - \mathcal{A}^\Delta(t, p_0, \mathbf{p})]. \quad (5)$$

It is important to stress that (5) does not rely on a choice of the integration range  $\Delta$  in the definition of (4). This is because the  $p_0$ -integration picks  $\delta(\delta t)$  up to realize  $t_1 = t_2$  after taking each derivative on  $t_1$  and  $t_2$ . Although the results should be the same regardless of  $\Delta$ , the physical picture becomes more vivid if we choose an appropriate value of  $\Delta$ . Using that  $\mathcal{D}^\Delta(t, p_0, \mathbf{p})$  and  $\mathcal{A}^\Delta(t, p_0, \mathbf{p})$  are even and odd functions of  $p_0$ , respectively, we can readily confirm that the contribution from anti-particles amounts to just the same answer, which should be guaranteed by the charge conservation.

Here we note that our formula (5) looks significantly different from the LSZ-type expression as used in [16, 19]. We need go back to (2) and adopt (2) as a *definition* of the annihilation operator at time  $t$ . Then, we can pick  $\hat{a}_{\text{out}}(\mathbf{p})$  at  $t = \infty$  up from the boundary if we integrate the  $t$ -derivative of (2) with respect to  $t$ . The  $t$ -derivative leads to  $[\partial_t + iE_{\text{out}}(\mathbf{p})]$  on  $\hat{\phi}(t, \mathbf{p})$ , so we get  $[\partial_t^2 + E_{\text{out}}^2(\mathbf{p})]$  as usual in the LSZ reduction formula. A problem in this argument is that  $E_{\text{out}}(\mathbf{p})$  makes sense only at  $t = \infty$  and there is no clear-cut prescription to identify the dispersion relation for any  $t$ . In the adiabatic limit with slowly changing vector potential  $\mathbf{A}(t)$ , one may be able to approximate it as  $E(t, \mathbf{p}) = \sqrt{[\mathbf{p} + e\mathbf{A}(t)]^2 + m^2}$  as assumed in [27, 28] (see also [29] for more discussions on the physical interpretation). Thus, with this subtlety about the dispersion relation at intermediate time, we do not think that the LSZ-type expression is any more advantageous than our formula (5).

Only for completeness let us make a remark on another expression with use of Green's functions. We can perform the Wigner transform for the retarded and advanced propagators to define  $D_R^\Delta(t, p_0, \mathbf{p})$  and  $D_A^\Delta(t, p_0, \mathbf{p})$  as well as the Feynman (time-ordered) propagator,  $D_F^\Delta(t, p_0, \mathbf{p})$ . Then, (5) is just equivalent with

$$\frac{dN}{d^3\mathbf{p}} = \lim_{t \rightarrow \infty} \frac{V}{(2\pi)^3} \int \frac{dp_0}{2\pi} \frac{1}{2E_{\text{out}}(\mathbf{p})} \left[ \frac{1}{4} \partial_t^2 + (p_0 + E_{\text{out}}(\mathbf{p}))^2 \right] [D_F^\Delta(t, p_0, \mathbf{p}) - D_R^\Delta(t, p_0, \mathbf{p})]. \quad (6)$$

This expression might be more tractable if one wants to apply the Schwinger-Keldysh formalism to compute the correlation functions.

To gain some feeling about how our formula works, we will take a quick look at the typical behaviour of these spectral functions in the asymptotic states where we can expand the field in terms of plane waves.

### 3. Spectral functions in the asymptotic states

Because the dynamical spectral functions are less known objects than more conventional ones in equilibrated matter, we will devote this section to the exploration of how they look like in the asymptotic in- and out-states. To reduce unnecessary complication, we shall limit our discussion to the choice of  $\hat{\rho}_{\text{in}} = |0_{\text{in}}\rangle\langle 0_{\text{in}}|$  for the moment. We will address an extension to the finite temperature/density environment in the later section. We denote the annihilation operators,  $\hat{a}_{\text{in}}(\mathbf{p})$  and  $\hat{b}_{\text{in}}(\mathbf{p})$ , respectively, for particles and anti-particles, with which the vacuum  $|0_{\text{in}}\rangle$  is defined. For a practical purpose we take the range of  $t$  from  $-T$  to  $T$  with a sufficiently large  $T$ .

The field operator in the in-state around  $t = -T$  is then a superposition of the plane waves with  $\hat{a}_{\text{in}}(\mathbf{p})$  for the positive-energy oscillation and  $\hat{b}_{\text{in}}^\dagger(\mathbf{p})$  for the negative-energy oscillation, i.e.

$$\hat{\phi}(t \sim -T, \mathbf{p}) = \frac{1}{\sqrt{2E_{\text{in}}(\mathbf{p})}} \left[ \hat{a}_{\text{in}}(\mathbf{p}) e^{-iE_{\text{in}}(\mathbf{p})t} + \hat{b}_{\text{in}}^\dagger(-\mathbf{p}) e^{iE_{\text{in}}(\mathbf{p})t} \right]. \quad (7)$$

The plane waves should be the solutions of the equation of motion around  $t = -T$ , and they evolve to a mixture of the positive- and the negative-energy states as  $t$  elapses toward the interacting region. We can then parametrise this mixing effect as

$$\frac{e^{-iE_{\text{in}}(\mathbf{p})t}}{\sqrt{2E_{\text{in}}(\mathbf{p})}} (t \sim -T) \longrightarrow \frac{\alpha_{\mathbf{p}} e^{-iE_{\text{out}}(\mathbf{p})t} + \beta_{\mathbf{p}}^* e^{iE_{\text{out}}(\mathbf{p})t}}{\sqrt{2E_{\text{out}}(\mathbf{p})}} (t \sim T), \quad (8)$$

where the Bogoliubov coefficients,  $\alpha_{\mathbf{p}}$  and  $\beta_{\mathbf{p}}$ , are determined according to the equation of motion, and a similar relation should hold for another branch of solution starting with  $\propto e^{iE_{\text{in}}(\mathbf{p})t}$ . In fact, if the Hamiltonian is Hermite, the complex conjugate of the above relation is true, so the field operator in the out-state at  $t = T$  then reads,

$$\begin{aligned} \hat{\phi}(t \sim \infty, \mathbf{p}) = \frac{1}{\sqrt{2E_{\text{out}}(\mathbf{p})}} \Big\{ & [\alpha_{\mathbf{p}} \hat{a}_{\text{in}}(\mathbf{p}) + \beta_{\mathbf{p}} \hat{b}_{\text{in}}^\dagger(-\mathbf{p})] e^{-iE_{\text{out}}(\mathbf{p})t} \\ & + [\alpha_{\mathbf{p}}^* \hat{b}_{\text{in}}^\dagger(-\mathbf{p}) + \beta_{\mathbf{p}}^* \hat{a}_{\text{in}}(\mathbf{p})] e^{iE_{\text{out}}(\mathbf{p})t} \Big\}. \quad (9) \end{aligned}$$

This defines the creation/annihilation operators in the out-state, and by requiring the canonical commutation relation for them, we can find the normalisation condition,  $|\alpha_{\mathbf{p}}|^2 - |\beta_{\mathbf{p}}|^2 = 1$ .

At this point we can immediately recover the known result for the Schwinger mechanism directly from (3). Applying the operator  $[\partial_t - iE_{\text{out}}(\mathbf{p})]$  on  $\hat{\phi}$  we project the positive-energy part out, and then we can plug the number operator of (3) into (1), which yields an estimate of produced particles as

$$\frac{dN}{d^3\mathbf{p}} = \frac{1}{(2\pi)^3} \langle 0_{\text{in}} | \beta_{\mathbf{p}}^* \hat{b}_{\text{in}}(-\mathbf{p}) \beta_{\mathbf{p}} \hat{b}_{\text{in}}^\dagger(-\mathbf{p}) | 0_{\text{in}} \rangle = \frac{V|\beta_{\mathbf{p}}|^2}{(2\pi)^3}. \quad (10)$$

This is a standard formula for the particle production obtained via the Bogoliubov transformation [30]. Now it is intriguing to check how our formula (5) gives rise to the same answer.

We can immediately compute the spectral functions from the asymptotic forms (7) and (9) if we take  $T \gg \Delta$ . Then, the spectral functions at  $t = -T$  have no access to the region with non-vanishing background fields, so they take a familiar expression just for non-interacting particles;

$$\mathcal{A}^\Delta(t \sim -T, p_0, \mathbf{p}) = \frac{\pi}{E_{\text{in}}(\mathbf{p})} \left[ \delta(p_0 - E_{\text{in}}(\mathbf{p})) - \delta(p_0 + E_{\text{in}}(\mathbf{p})) \right], \quad (11)$$

$$\mathcal{D}^\Delta(t \sim -T, p_0, \mathbf{p}) = \frac{\pi}{E_{\text{in}}(\mathbf{p})} \left[ \delta(p_0 - E_{\text{in}}(\mathbf{p})) + \delta(p_0 + E_{\text{in}}(\mathbf{p})) \right]. \quad (12)$$

In this case  $\mathcal{D}^\Delta - \mathcal{A}^\Delta$  has only a term that is proportional to  $\delta(p_0 + E_{\text{in}}(\mathbf{p}))$ , and thus the produced particle is vanishing as is obvious from (5). Now let us go into later time when these functions should change their shape. Once (9) eventually follows, it is just a simple arithmetic procedure to reach,

$$\mathcal{A}^\Delta(t \sim T, p_0, \mathbf{p}) = \frac{\pi}{E_{\text{out}}(\mathbf{p})} \left[ \delta(p_0 - E_{\text{out}}(\mathbf{p})) - \delta(p_0 + E_{\text{out}}(\mathbf{p})) \right], \quad (13)$$

$$\begin{aligned} \mathcal{D}^\Delta(t \sim T, p_0, \mathbf{p}) = & \left[ |\alpha_{\mathbf{p}}|^2 + |\beta_{\mathbf{p}}|^2 \right] \frac{\pi}{E_{\text{out}}(\mathbf{p})} \left[ \delta(p_0 - E_{\text{out}}(\mathbf{p})) + \delta(p_0 + E_{\text{out}}(\mathbf{p})) \right] \\ & + \frac{2}{E_{\text{out}}(\mathbf{p})} \text{Re} \left[ \alpha_{\mathbf{p}} \beta_{\mathbf{p}} e^{-2iE_{\text{out}}(\mathbf{p})t} \right] 2\pi \delta(p_0). \end{aligned} \quad (14)$$

Here, again, we required  $T \gg \Delta$ . There are two interesting observations as perceived from the above: (1) Pauli-Jordan's function  $\mathcal{A}^\Delta$  is insensitive to the Bogoliubov transformation and the overall factor is  $|\alpha_{\mathbf{p}}|^2 - |\beta_{\mathbf{p}}|^2 = 1$ . (2) Hadamard's function  $\mathcal{D}^\Delta$  is affected by the mixing effect by  $|\alpha_{\mathbf{p}}|^2 + |\beta_{\mathbf{p}}|^2 \neq 1$  and, besides, an interference term  $\propto \alpha_{\mathbf{p}} \beta_{\mathbf{p}}$  appears. We emphasise that such an interference term is usually absent and is quite peculiar to the Bogoliubov mixing effect.

Then, the difference between these two spectral functions consists of three terms as follows,

$$\begin{aligned} \mathcal{D}^\Delta - \mathcal{A}^\Delta = & |\beta_{\mathbf{p}}|^2 \frac{2\pi}{E_{\text{out}}(\mathbf{p})} \delta(p_0 - E_{\text{out}}(\mathbf{p})) + |\alpha_{\mathbf{p}}|^2 \frac{2\pi}{E_{\text{out}}(\mathbf{p})} \delta(p_0 + E_{\text{out}}(\mathbf{p})) \\ & + \frac{2}{E_{\text{out}}(\mathbf{p})} \text{Re} [\alpha_{\mathbf{p}} \beta_{\mathbf{p}} e^{-2iE_{\text{out}}(\mathbf{p})t}] 2\pi \delta(p_0). \end{aligned} \quad (15)$$



We can make it sure by the explicit calculation that the second ( $\propto |\alpha_{\mathbf{p}}|^2$ ) and the third ( $\propto \alpha_{\mathbf{p}}\beta_{\mathbf{p}}$ ) terms have no finite contribution if applied to (5), and only the first ( $\propto |\beta_{\mathbf{p}}|^2$ ) term is relevant to the particle production, which yields exactly the same answer as (10).

Although the calculations are very easy, the expression of (15) in the out-state has profound implications. In many situations we typically have  $\alpha_{\mathbf{p}} \approx 1$  and  $|\beta_{\mathbf{p}}| \ll 1$ , for which the first term is much smaller than the third interference term. In the next section, indeed, we will numerically compute the spectral functions and confirm that this is the case. It would be an interesting future problem to think of a way to make use of the interference term in order to probe the Bogoliubov mixing effect experimentally.

Before closing this section, it would be instructive to understand how the standard propagators are modified by the Bogoliubov transformation. Surprisingly, we find that the retarded propagator is intact under the mixing effect and only the Feynman propagator depends on the Bogoliubov coefficients. That is,

$$D_R^\Delta(t \sim T, p_0, \mathbf{p}) = P \frac{i}{p_0^2 - E_{\text{out}}^2(\mathbf{p})} + \frac{\pi}{2E_{\text{out}}(\mathbf{p})} \left[ \delta(p_0 - E_{\text{out}}(\mathbf{p})) - \delta(p_0 + E_{\text{out}}(\mathbf{p})) \right], \quad (16)$$

$$D_F^\Delta(t \sim T, p_0, \mathbf{p}) = P \frac{i}{p_0^2 - E_{\text{out}}^2(\mathbf{p})} + (|\alpha_{\mathbf{p}}|^2 + |\beta_{\mathbf{p}}|^2) \frac{\pi}{2E_{\text{out}}(\mathbf{p})} \left[ \delta(p_0 - E_{\text{out}}(\mathbf{p})) + \delta(p_0 + E_{\text{out}}(\mathbf{p})) \right] \\ + \frac{1}{E_{\text{out}}(\mathbf{p})} \text{Re}(\alpha_{\mathbf{p}}\beta_{\mathbf{p}}e^{-2iE_{\text{out}}(\mathbf{p})t}) 2\pi\delta(p_0), \quad (17)$$

where  $P$  stands for taking the principal value. It is clear at glance that  $\mathcal{D}^\Delta - \mathcal{A}^\Delta = 2(D_F^\Delta - D_R^\Delta)$  holds as it should.

Now it is time to take one step forward to understand how the spectral functions should evolve continuously from the in-state to the out-state as a function of  $t$ . In the aim of visualising the behaviour with increasing  $t$ , we need to perform numerical calculations. In the next section we present our numerical results.

#### 4. Spectral functions in pulsed electric fields

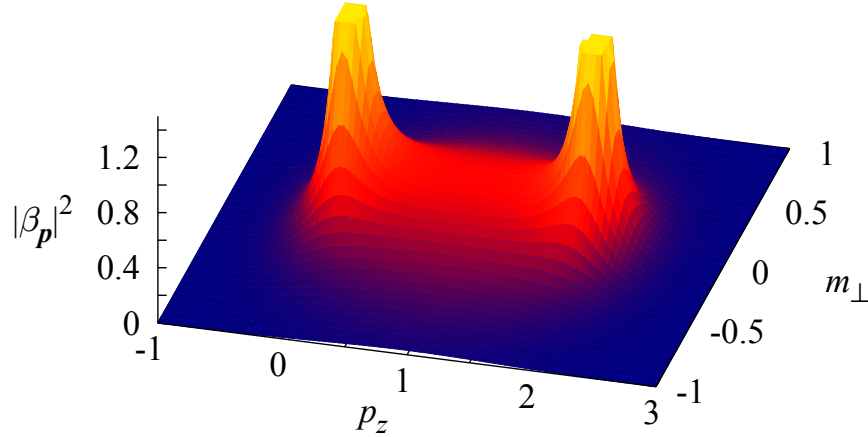
We here solve the equation of motion for given electric fields. In principle, we can numerically deal with arbitrary electric fields within our present approximation to neglect the back-reaction. Although the analytical solution is not demanded here, we shall choose one of the most well-investigated profile known as the Sauter potential [31], which is solvable and identifiable with a pulsed electric field,

$$E(t) = E \text{sech}^2(\omega t). \quad (18)$$

The frequency parameter  $\omega$  characterises the life time of the applied electric field. Let us choose the  $z$  axis along the direction of the electric field, and then the associated vector potential reads,

$$A_z(x) = \frac{E}{\omega} [\tanh(\omega t) - 1]. \quad (19)$$





**Figure 1.** Bogoliubov coefficient  $|\beta_{\mathbf{p}}|^2$  for the choice of  $\omega/\sqrt{eE} = 1$ . All quantities are measured in unit of  $\sqrt{eE}$ .

Then, we can find two independent solutions,  $\psi_p^{(\pm)}(t)$ , by solving the equation of motion under this vector potential,

$$\left[ \partial_t^2 + \left( p_z + \frac{eE}{\omega} [\tanh(\omega t) - 1] \right)^2 + m_\perp^2 \right] \psi_p^{(\pm)}(t) = 0, \quad (20)$$

where  $m_\perp$  represents the transverse mass,  $m_\perp^2 := p_x^2 + p_y^2 + m^2$ . We should impose the following boundary conditions;

$$\psi_p^{(\pm)}(t = -T) = \frac{1}{\sqrt{2E_{\text{in}}(\mathbf{p})}} e^{\mp i E_{\text{in}}(\mathbf{p})t}, \quad (21)$$

for large enough  $T$ , with the dispersion relations,

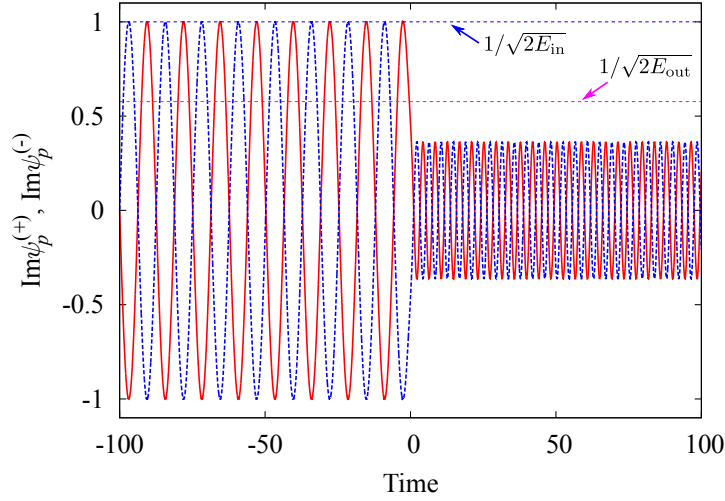
$$E_{\text{in}}(\mathbf{p}) = \sqrt{(p_z - 2eE/\omega)^2 + m_\perp^2}, \quad E_{\text{out}}(\mathbf{p}) = \sqrt{p_z^2 + m_\perp^2}. \quad (22)$$

One can write the analytical expressions of  $\psi_p^{(\pm)}(t)$  using the hyper-geometric functions. Therefore, the number of produced particle or  $|\beta_{\mathbf{p}}|^2$  is analytically known. Hereafter we shall refer to all quantities with mass dimensions in unit of the electric field  $\sqrt{eE}$  and present our results with dimensionless numbers. In this work we work with a specific choice of

$$\frac{\omega}{\sqrt{eE}} = 1, \quad (23)$$

to investigate the effect of pulsed electric fields. A different choice of  $\omega$  makes no qualitative change in our resulting spectral functions. We make a plot in figure 1 to show the analytical structure of  $|\beta_{\mathbf{p}}|^2$  as a function of the longitudinal momentum  $p_z$  and the transverse mass  $m_\perp$ .

Obviously from figure 1, the particle number becomes greater for smaller  $m_\perp$ . The produced particles are accelerated to the positive  $z$  direction by the electric field, and as understood from (20),  $p_z$  is shifted by  $0 \sim 2eE/\omega$  during the time evolution. This means that the momentum distribution of the produced particles should spread over  $p_z = 0 \sim 2eE/\omega$ . We can confirm this expectation explicitly on figure 1.



**Figure 2.** Two independent wave functions satisfying the given boundary conditions (21) at  $t = -T$  with  $T = 100$ .

To discuss the effect of the particle production in a reasonably visible manner, we will look at the point of following momenta,

$$\frac{m_{\perp}}{\sqrt{eE}} = 0, \quad \frac{p_z}{\sqrt{eE}} = \frac{\sqrt{eE}}{\omega} = 1.5, \quad (24)$$

which deviates from the pronounced peak seen in figure 1. One might have thought that the exact peak position would be a better choice, but if we choose  $m_{\perp} = 0$  and  $p_z = 2$ , for instance, the numerical calculations result in singularity out of control.

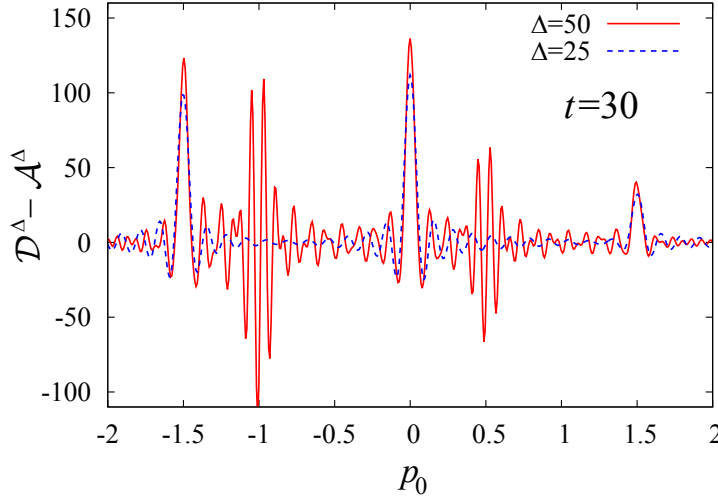
With these parameters we solve the equation of motion (20) numerically to find  $\psi_p^{(\pm)}(t)$ , the results of which are shown in figure 2. We used the 4th-order Runge-Kutta (RK4) method and took 20000 points to discretise along the time direction. We imposed the boundary conditions (21) at  $t = -T$  with  $T = 100$ . Because the equation of motion is real,  $\psi_p^{(-)}(t) = \psi_p^{(+)*}(t)$  follows immediately, and this means that the real part of them should be identical. This is why we present the imaginary part in figure 2, and indeed, we can make it sure that our numerical calculations go correctly to respect  $\text{Im}\psi_p^{(-)}(t) = -\text{Im}\psi_p^{(+)}(t)$ .

The in-state around  $t \sim -T$  has the field amplitude of the correct normalisation  $1/\sqrt{2E_{\text{in}}(\mathbf{p})}$ , while at later time, as seen in figure 2, the amplitude deviates from  $1/\sqrt{2E_{\text{out}}(\mathbf{p})}$ . This discrepancy is attributed to the mixing between the positive- and the negative-energy states and thus signals for the Bogoliubov transformation.

Once we have the wave-functions, we can construct the spectral functions for any  $t$ , i.e. a simple calculation leads to

$$\mathcal{A}^{\Delta}(t, p_0, \mathbf{p}) = \int_{-\Delta}^{\Delta} d\delta t e^{ip_0\delta t} \left[ \psi_{\mathbf{p}}^{(+)}(t + \tfrac{1}{2}\delta t) \psi_{\mathbf{p}}^{(+)*}(t - \tfrac{1}{2}\delta t) - \psi_{\mathbf{p}}^{(-)}(t + \tfrac{1}{2}\delta t) \psi_{\mathbf{p}}^{(-)*}(t - \tfrac{1}{2}\delta t) \right], \quad (25)$$

for Pauli-Jordan's function and we can find a similar expression for Hadamard's function. The time evolution of the spectral functions may have dependence on the choice of  $\Delta$ .



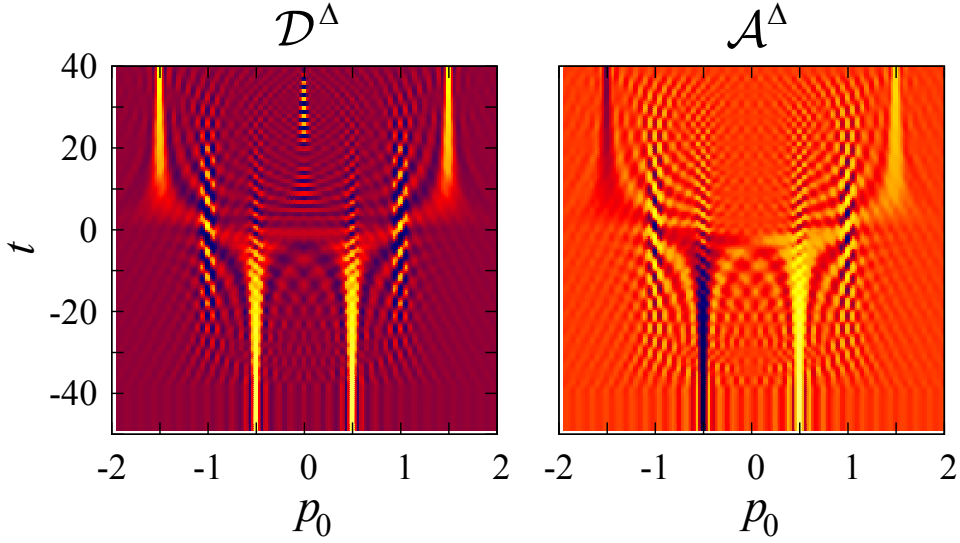
**Figure 3.**  $\Delta$  dependence of the spectral function  $\mathcal{D}^\Delta - \mathcal{A}^\Delta$  at  $t = 30$ . The solid (and dashed) curve represents the results at  $\Delta = 50$  (and 25, respectively).

Intuitively,  $\Delta$  corresponds to the observation time, as we already mentioned, to detect the quasi-particle behaviour in the oscillation pattern. For the concrete demonstration, let us take a look at figure 2 again; the temporal oscillation shows a constant pattern except near the origin where the system is disturbed by pulsed electric fields. So, around  $t = 30$  for example, if  $\Delta$  is less than 30, the quasi-particle behaviour is well separated from the interaction region at the origin and the spectral functions should be close to (15) then. If  $\Delta$  is greater than 30, however, the integration region covers the pulsed electric fields, which should alter the spectral shape. Indeed, as we can see in figure 3, we can confirm this anticipation by comparing the results at  $\Delta = 25 < t = 30$  and  $\Delta = 50$  for  $\mathcal{D}^\Delta - \mathcal{A}^\Delta$ .

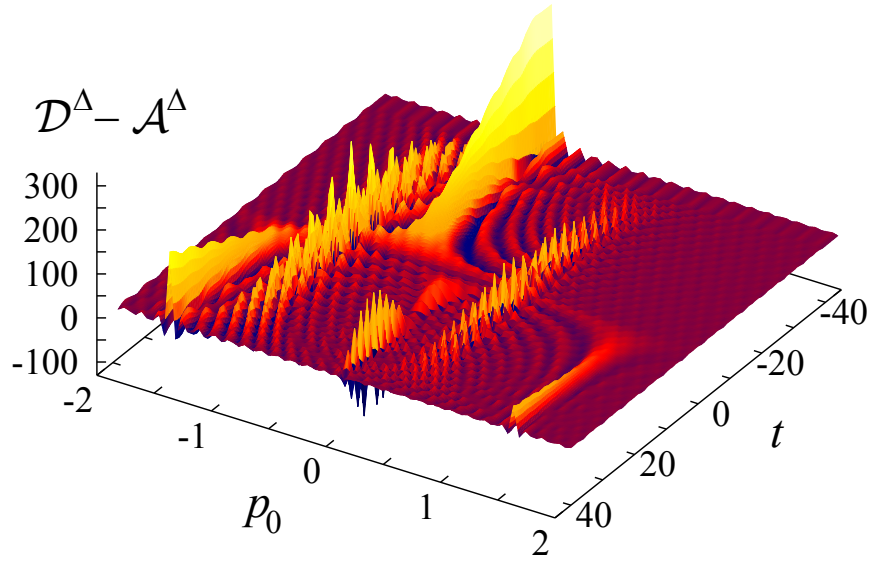
Figure 3 already indicates the Schwinger process of the particle production. We can see a peak at  $p_0 \sim E_{\text{out}}(\mathbf{p}) = 1.5$  and its height corresponds to  $|\beta_{\mathbf{p}}|^2$  according to (5). Precisely speaking, if we take  $\Delta \rightarrow \infty$ , the peak becomes a Dirac's delta function and the  $p_0$ -integration in (5) has a contribution from a point  $p_0 = E_{\text{out}}(\mathbf{p})$  only. Now that we implement the Wigner transformation with a finite  $\Delta$ , the peak is broadened and we should keep the  $p_0$ -integration over the range of the order of  $1/\Delta$ .

From (15) we can understand two more peaks are expected at  $p_0 \sim -E_{\text{out}}(\mathbf{p}) = -1.5$  and  $p_0 \sim 0$  which are evident in figure 3. For  $\Delta = 50$ , we are also aware of some enhancement around  $p_0 \sim -1$  and 0.5, which is quite non-trivial. They arise from the effect of the background fields when  $\Delta$  is comparable to or greater than  $t$ .

We are interested in the full temporal profile of these enhanced regions, so we make a density plot for  $\mathcal{D}^\Delta$  and  $\mathcal{A}^\Delta$  individually, as is shown in figure 4. This figure provides us with useful messages about the flow of the spectral peaks. First of all, the oscillatory peak at  $p_0 = 0$  appears only in  $\mathcal{D}^\Delta$  at late  $t$ , as is the case in the out-state (14). Second, we can notice that the intermediate enhancement around  $p_0 \sim -1$  and 0.5 emerges in both  $\mathcal{D}^\Delta$  and  $\mathcal{A}^\Delta$ . Because the enhancement originates from time-dependent



**Figure 4.** Density plots of the spectral functions,  $\mathcal{D}^\Delta$  and  $\mathcal{A}^\Delta$ , for the choice of (23) and (24) and  $\Delta = 40$ . The bright (and dark) colour indicates a larger (and smaller, respectively) value.



**Figure 5.** Evolution of the spectral function difference  $\mathcal{D}^\Delta - \mathcal{A}^\Delta$  for the choice of (23) and (24) with  $\Delta = 40$ .

background fields, it would be conceivable that no simple pattern but complicated time dependence may well occur. This is not the case, however, and the enhancement goes rather straight in time. We can observe this in a clearer way in the form of not the density plot but the three-dimensional (3D) plot as presented in figure 5.

In figure 5 we plot  $\mathcal{D}^\Delta - \mathcal{A}^\Delta$  that is the difference between two in figure 4. Near the in-state, as seen in figure 5, there stands only one peak in the vicinity of  $p_0 = -E_{\text{in}}(\mathbf{p})$ , which agrees perfectly with the asymptotic analysis (12). Then, this peak diminishes

with increasing time, and meanwhile, the intermediate oscillatory modes grow up at  $p_0 \sim -1$  and  $0.5$ . Eventually, these modes fade away, and at the same time, the spectral function approaches the asymptotic form of (15). At late time we can recognise a small peak around  $p_0 \sim 1.5$  and this peak amounts to the Schwinger mechanism. In other words, the Schwinger mechanism is a phenomenon that takes account of such a small portion of the whole spectral shape. It is certainly worth considering other tools to diagnose a wider region of the spectral functions, which is an interesting future problem beyond the present scope.

## 5. Extension to the finite temperature

Finally we shall extend our analysis to a more general situation of the initial state. A more non-trivial but still controllable example is the finite temperature/density calculation of the Schwinger mechanism [32]. Let us assume that the initial state is a mixed state characterised by the following density matrix,

$$\hat{\rho}_\infty = \frac{\exp[-\beta E_{\text{in}}(\mathbf{p}) \hat{a}_{\text{in}}^\dagger(\mathbf{p}) \hat{a}_{\text{in}}(\mathbf{p})]}{\text{tr}\{\exp[-\beta E_{\text{in}}(\mathbf{p}) \hat{a}_{\text{in}}^\dagger(\mathbf{p}) \hat{a}_{\text{in}}(\mathbf{p})]\}} , \quad (26)$$

with  $\beta$  being the inverse temperature. Then the straightforward calculation immediately leads to the produced particle number given as

$$\frac{dN}{d^3\mathbf{p}} - \frac{V f_{\mathbf{p}}}{(2\pi)^3} = \frac{V |\beta_{\mathbf{p}}|^2}{(2\pi)^3} \left(1 + f_{\mathbf{p}} + \bar{f}_{-\mathbf{p}}\right) , \quad (27)$$

where  $f_{\mathbf{p}}$  and  $\bar{f}_{\mathbf{p}}$  represent the Bose-Einstein distribution function, respectively, for the particle and the anti-particle with the in-state energy  $E_{\text{in}}(\mathbf{p})$ , namely,

$$f_{\mathbf{p}} := \frac{1}{e^{\beta[E_{\text{in}}(\mathbf{p}) - \mu]} - 1} , \quad \bar{f}_{\mathbf{p}} := \frac{1}{e^{\beta[E_{\text{in}}(\mathbf{p}) + \mu]} - 1} \quad (28)$$

with a chemical potential  $\mu$  introduced. We note that, in the left-hand side of (27), the number of thermal particles is subtracted since they are irrelevant to the particle production. This result is understandable also from the spectral functions. We can find that Hadamard's function picks the Bose-Einstein distribution function up as

$$\begin{aligned} \mathcal{D}^\Delta(t, p_0, \mathbf{p}) = \int_{-\Delta}^{\Delta} d\delta t e^{ip_0 \delta t} \Big[ & \psi_{\mathbf{p}}^{(+)}(t + \tfrac{1}{2}\delta t) \psi_{\mathbf{p}}^{(+)*}(t - \tfrac{1}{2}\delta t) (1 + 2f_{\mathbf{p}}) \\ & + \psi_{\mathbf{p}}^{(-)}(t + \tfrac{1}{2}\delta t) \psi_{\mathbf{p}}^{(-)*}(t - \tfrac{1}{2}\delta t) (1 + 2\bar{f}_{-\mathbf{p}}) \Big] , \quad (29) \end{aligned}$$

but Pauli-Jordan's function  $\mathcal{A}^\Delta$  remains independent of the medium effect and is not changed from the vacuum expression (25). Such a qualitative difference between  $\mathcal{D}^\Delta$  and  $\mathcal{A}^\Delta$  makes a sharp contrast and is not quite comprehensible on the intuitive level.

The particle production is increased by the Bose-Einstein distribution function. Therefore, if  $f_{\mathbf{p}}$  takes a macroscopic value, this increase must be a sizable effect. In the high- $T$  limit, in fact, the distribution function approaches,  $f_{\mathbf{p}} \sim \bar{f}_{\mathbf{p}} \rightarrow T/E_{\text{in}}(\mathbf{p})$ , and the particle production is significantly enhanced by  $2T/E_{\text{in}}(\mathbf{p})$ , which is a substantial factor if  $T$  is large or  $E_{\text{in}}(\mathbf{p})$  is small enough, as is the case in the quark-gluon plasma. However,

the in-state already contains as many particles as  $f_{\mathbf{p}}$  and so the particle production is not practically enhanced if measured relative to the number of particles in the in-state.

Another interesting limit lies in a finite chemical potential that makes  $\bar{f}_{\mathbf{p}}$  (or  $f_{\mathbf{p}}$ ) be much bigger than  $f_{\mathbf{p}}$  (or  $\bar{f}_{\mathbf{p}}$ ). This may well opens a new possibility for the experimental detection of the Schwinger process. For example, if we have a macroscopic occupation number like the Bose-Einstein condensate of scalar particles (that is actually a superconductor), the anti-particle (hole) that did not exist in the in-state is produced with a gigantic enhancement factor by the macroscopic occupation number. It may be worth pursuing this possibility further in the future research.

## 6. Conclusions

We found a useful formula that relates the particle production to the dynamical (time-dependent) spectral functions. We then clarified the basic properties of these spectral functions and proceeded to the numerical calculation of the spectral functions using the solutions of the equation of motion for a complex scalar field theory under pulsed electric fields. We closely studied the time evolution of the spectral functions. Wigner transformed Hadamard's function turned out to exhibit an oscillatory mode at  $p_0 = 0$  as a result of the Bogoliubov mixing. This peak is larger by one power of the Bogoliubov coefficient as compared to the other peak corresponding to the Schwinger mechanism. This structure hints a new possibility of measurement that verifies the Bogoliubov mixing. Another non-trivial finding is the appearance of transient enhancement in the intermediate time region. In spite of time-dependent background fields, the enhancement occurs somehow in an organised manner, which implies that some unknown mechanism underlies the real-time dynamics. We also extended our discussions to the finite temperature/density case to identify the medium enhancement factor.

Apart from the Schwinger problem, from a more general perspective of theoretical physics, the dynamical properties of the spectral functions are quite non-trivial and are still less unknown than the equilibrated matter. We emphasise that these spectral functions are essential ingredients to think of real-time physics, and the particle production is actually one of the possible applications. In this sense we should continuously invest our efforts to deepen the theoretical understanding of the dynamical spectral functions and the present work should contribute to the first step along this direction.

## Acknowledgments

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